A Two-Stage Modeling and Solution Framework for Multisite Midterm Planning under Demand Uncertainty

Anshuman Gupta and Costas D. Maranas*

Department of Chemical Engineering, The Pennsylvania State University, University Park, Pennsylvania 16802

A two-stage, stochastic programming approach is proposed for incorporating demand uncertainty in multisite midterm supply-chain planning problems. In this bilevel decision-making framework, the *production* decisions are made "here-and-now" prior to the resolution of uncertainty, while the *supply-chain* decisions are postponed in a "wait-and-see" mode. The challenge associated with the expectation evaluation of the inner optimization problem is resolved by obtaining its closed-form solution using linear programming (LP) duality. At the expense of imposing the normality assumption for the stochastic product demands, the evaluation of the expected second-stage costs is achieved by analytical integration yielding an equivalent convex mixed-integer nonlinear problem (MINLP). Computational requirements for the proposed methodology are shown to be much smaller than those for Monte Carlo sampling. In addition, the cost savings achieved by modeling uncertainty at the planning stage are quantified on the basis of a rolling horizon simulation study.

Introduction and Motivation

Supply-chain planning is concerned with the coordination and integration of key business activities undertaken by an enterprise, from the procurement of raw materials to the distribution of the final products to the customers. In the prevailing volatile business environment, with ever changing market conditions and customer expectations, it is necessary to consider the impact of uncertainties involved in the supply chain. Sources of uncertainty in production planning can be categorized as short-term or long-term based on their time frames.[1] Short-term uncertainty may include day-to-day processing variations, canceled/rushed orders, equipment failure, etc. Long-term uncertainty refers to raw material/final product unit price fluctuations, demand variations, and production rate changes occurring over longer time frames.[2] It is the latter type of uncertainty that is addressed in this work.

One of the key sources of uncertainty in any production−distribution system is the product demand. Product demand fluctuations over medium-term (1−2 years) to long-term (5−10 years) planning horizons may be significant. Deterministic planning and scheduling models may thus yield unrealistic results by failing to capture the effect of demand variability on the tradeoff between lost sales and inventory holding costs. Failure to incorporate a stochastic description of the product demand could lead to either unsatisfied customer demand and loss of market share or excessively high inventory holding costs.[3]

Recognition of this drawback of deterministic models has led to a number of publications devoted to studying process planning under uncertainty. Some of the key aspects that have been addressed are design and operation of batch plants,[1,3,4−8] issues concerning flexibility and reliability in process design,[9−11] and long-range planning and capacity expansion of chemical process networks.[8,12−14] As is evident from the literature reviewed above, almost all research has been limited to (i) batch processing systems and (ii) single production sites. Key features such as the presence of (semi)-continuous processes and multiple production sites have, so far, not been considered in detail. In view of this, the incorporation of demand uncertainty in (semi)-continuous, midterm, multisite planning is addressed in this work. To this end, the deterministic midterm planning model of McDonald and Karimi[15] is adopted as the benchmark formulation.

Modeling and Decision Making under Uncertainty

A key component of decision making under uncertainty is the representation of the stochastic parameters. Two distinct ways of representing uncertainty exist. The scenario-based approach[1,7,16] attempts to represent a random parameter by forecasting all its possible future outcomes. The main drawback of this technique is that the number of scenarios increases exponentially with the number of uncertain parameters, leading to an exponential increase in the problem size. To circumvent this difficulty, continuous probability distributions for the random parameters are frequently used.[6,8,17] At the expense of introducing nonlinearities into the problem through multivariate integration over the continuous probability space, a substantial decrease in the size of the problem is usually achieved. In this work, the latter approach is used for describing uncertainty. The product demands are modeled as normally distributed random variables. This approach has been widely invoked in the literature[6,17,18] as it captures the essential features of demand uncertainty and is convenient to use.

One of the most widely used techniques for decision making under uncertainty is two-stage stochastic programming.[12,14,17,19−25] In this technique, the decision variables of the problem are partitioned into two sets. The *first-stage* variables, also known as *design* variables, correspond to those decisions that need to be made prior to resolution of uncertainty ("here-and-now" decisions). Subsequently, based on these decisions and the realiza-

* Author to whom all correspondence should be addressed. E-mail: costas@psu.edu. Phone: 814-863-9958. Fax: 814-865-7846.

10.1021/ie9909284 CCC: $19.00
Published on Web 10/02/2000

tion of the random events, the *second-stage* or *control* decisions are made subject to the restrictions of the second-stage *recourse* problem ("wait-and-see" decisions). The presence of uncertainty is translated into the stochastic nature of the costs associated with the second-stage decisions. Therefore, the objective function consists of the sum of the first-stage decision costs and the *expected* second-stage recourse costs.

The main challenge associated with solving two-stage stochastic problems is the evaluation of the expectation of the inner recourse problem. For the scenario-based representation of uncertainty, this can be achieved by explicitly associating a second-stage variable with each scenario and solving the large-scale *extensive* formulation[26] by efficient solution techniques such as Dantzig–Wolfe decomposition[27] and Benders decomposition.[28] For continuous probability distributions, this challenge has been primarily resolved through the explicit/implicit discretization of the probability space for approximating the multivariate probability integrals. The two most commonly used discretization strategies in the chemical engineering literature are Monte Carlo sampling[16,29] and Gaussian quadrature.[8,9,30,31] The key advantage of these methods lies in the fact that they are largely insensitive to the type of probability distribution. The main disadvantage, as in the scenario-based approach, is the sharp increase in computational requirements with increasing numbers of uncertain parameters.[2]

Thus far, applications of stochastic planning models have been limited compared to those of deterministic models because of their computationally intensive nature. One of the first attempts at narrowing this computational gap is the work of Petkov and Maranas.[17] In this work, the problem of designing single-product campaign batch plants under demand uncertainty is addressed. By the explicit solution of the inner problem followed by analytical integration over all product demand realizations, the need for discretization of the probability space is obviated. The stochastic attributes of the problem are translated into an equivalent deterministic optimization problem at the expense of introducing nonlinearities into the problem. The proposed technique is shown to result in significant savings in computational requirements over quadrature integration. A similar treatment of uncertainty for the more complex midterm supply-chain planning problem is pursued in this work.

The rest of the paper is organized as follows. In the next section, a two-stage stochastic formulation for incorporating demand uncertainty in the midterm planning problem posed by McDonald and Karimi[15] is proposed. First, the special case of a *single* production site is addressed to motivate the proposed solution methodology. Subsequently, utilizing the insight gained from the analysis of the single-site case, the more general *multisite* setting is discussed. A motivating example highlighting the key features of the proposed solution methodology is then presented, followed by the computational results for two larger examples. Finally, the work is summarized, and concluding remarks are provided.

Two-Stage Stochastic Formulation

The midterm production planning model of McDonald and Karimi[15] forms the basis of this work. The key tradeoff captured by this model, in the spirit of the classic multi-item capacitated lot-sizing problem, is that between fixed setup cost and inventory holding cost. The model utilizes the concept of "slots" within which the various supply-chain activities are assumed to occur. The duration of this slot ranges from 1 to 2 months, in accordance with the midterm nature of the model. Further details about the various model attributes and limitations can be found in McDonald and Karimi[15] and Gupta and Maranas.[32]

McDonald and Karimi[15] classify the constraints of the model into two distinct sets. The first set of *production constraints* ensure that an efficient allocation of the production capacity is achieved at the various production sites. These constraints determine the optimal operating policies at the production sites. The second category of constraints are referred to as *supply-chain constraints*. These constraints model the post-production activities of inventory management and effective allocation of customer demand. The idea of distinguishing between production constraints and supply-chain constraints naturally extends to the decision variables of the model,[32] resulting in the partitioning of the variables into *production variables* and *supply-chain variables*. The production variables, which establish the location and timing of production runs, length of campaigns, production amounts, and consumption of raw materials, uniquely define the production levels and resource utilization in the supply chain. The supply-chain variables determine the flow of materials throughout the production–distribution system while accounting for inventory management. Because of the considerable lead times involved in the production process, the production variables need to be set here-and-now, prior to demand realization. The supply-chain activities, such as inventory control and customer demand allocation, on the other hand, can be performed in a wait-and-see mode.

The classification of the variables and constraints of the midterm planning model into two distinct categories results in a two-stage hierarchical decision-making framework, which can be effectively utilized for incorporating demand uncertainty. The two-stage midterm planning model under demand uncertainty is formulated as

(2SMP)

$$\min_{\substack{P_{ijs}, RL_{ijs}, FRL_{fjs}\\ C_{is}, W_{iss'}, Y_{fjs}}} \sum_{f,j,s} FC_f Y_{fjs} + \sum_{i,j,s} v_{ijs} P_{ijs} + \sum_{i,s} p_{is} C_{is} + \sum_{i,s,s'} t_{iss'} W_{iss'}$$

$$+E_{\theta_i}\left[\begin{array}{ll} \displaystyle\min_{\substack{S_{is}, I_{is}\\ I^{\Delta}_{is}, I^{-}_{i}}} \sum_{i,s} t_{is} S_{is} + \sum_{i,s} \zeta_{is} I^{\Delta}_{is} + \sum_{i,s} h_{is} I_{is} + \sum_{i} \mu_i I^{-}_{i} & \\ \text{such that} & \\ \displaystyle\sum_{s} S_{is} \le \theta_i & \forall\ i \in I^{FP} \\ \displaystyle I_{is} = I^{0}_{is} + \sum_{j} P_{ijs} - \sum_{s'} W_{iss'} - S_{is} & \forall\ s, i \in I^{RM} \\ \displaystyle\theta_i - \sum_{s} S_{is} \le I^{-}_{i} \le \theta_i & \forall\ i \in I^{FP} \\ I^{L}_{is} - I_{is} \le I^{\Delta}_{is} \le I^{L}_{is} & \forall\ s, i \in I^{RM} \\ S_{is}, I_{is}, I^{-}_{i}, I^{\Delta}_{is} \ge 0 & \end{array}\right]$$

subject to

$$P_{ijs} = R_{ijs} RL_{ijs} \qquad \forall\, j, s, i \in I^{RM} \tag{1}$$

$$C_{is} = \sum_{i'} \beta_{i'is} \sum_{j} P_{i'js} \qquad \forall\, s, i \in I^{FP} \tag{2}$$

$$C_{is} = \sum_{s'} W_{is's} \qquad \forall\, s, i \in I^{IP} \tag{3}$$

$$FRL_{fjs} = \sum_{i:\lambda_{if}=1} RL_{ijs} \qquad \forall\, j, s, f \tag{4}$$

$$\sum_{f} FRL_{fjs} \le H_{js} \qquad \forall\, j, s \tag{5}$$

$$MRL_{fjs} Y_{fjs} \le FRL_{fjs} \le H_{js} Y_{fjs} \qquad \forall\, f, j, s \tag{6}$$

$$P_{ijs}, RL_{ijs}, FRL_{fjs}, C_{is}, W_{iss'} \ge 0,\ Y_{fjs} \in \{0, 1\}$$

The first-stage production decisions correspond to P_{ijs}, RL_{ijs}, FRL_{fjs}, C_{is}, $W_{iss'}$, and Y_{fjs} in model 2SMP. The first-stage decision-making process is represented by the outer optimization problem consisting of eqs 1–6, which are the production constraints. The objective function of 2SMP is composed of two terms. The first includes the costs incurred in the production stage. These are the fixed and variable costs of production, raw material charges, and the cost of shipping intermediate products between production facilities. The second term quantifies the expected costs of the inner inventory-management recourse problem. The constraints of this embedded optimization problem are referred to as the supply-chain constraints, and the decision variables involved, S_{is}, I_{is}, I_{is}^{Δ}, and I_i^-, constitute the supply-chain variables. These variables are akin to control variables as they can be fine-tuned to ensure optimality in the face of uncertainty. The inner problem, thus, identifies the values of the supply-chain variables that minimize the total supply-chain cost for a given set of values of the production variables and demand realizations (θ_i). The basic idea of the methodology proposed for solving model 2SMP is based on obtaining a closed-form solution of this inner problem. This process is discussed for the special case of a *single* production site in the next section.

Single Production Site

Before addressing the more relevant multisite instance of problem 2SMP, the simpler case of a single production site is considered. This analysis provides the insight based upon which the more general multisite setting will be addressed next.

Consider the inner inventory-management problem for the single-site case. Elimination of the inventory variable (I_i) and the customer shortage variable (I_i^-) and corresponding redundant constraints results in the following form for the inner problem. Note that the site index has been omitted for convenience.

(IP_{SPS})

$$\sum_i h_i A_i + \sum_i \mu_i \theta_i + \min_{I_i^{\Delta}, S_i} \sum_i \zeta_i I_i^{\Delta} - \sum_i (h_i + \mu_i - t_i) S_i$$

subject to

$$S_i \le \theta_i \tag{7}$$

$$S_i - I_i^{\Delta} \le A_i - I_i^L \tag{8}$$

$$S_i \le A_i \tag{9}$$

$$S_i, I_i^{\Delta} \ge 0 \tag{10}$$

where

$$A_i = I_i^0 + \sum_j P_{ij} \tag{11}$$

Given the values for S_i and I_i^{Δ} as obtained from problem IP_{SPS}, I_i and I_i^- can be calculated “off-line” as

$$I_i = A_i - S_i \text{ and } I_i^- = \theta_i - S_i \tag{12}$$

The constraints of problem IP_{SPS} can be described as follows. Equation 7 enforces no overstocking at the customer. Equation 8, along with the nonnegativity of the inventory deviation variable, ensures that the underpenalty cost is incurred only when the inventory level is below the target safety stock level. Equation 9 represents the nonnegativity of inventory held.

Two important features of IP_{SPS} that are useful in characterizing its optimal solution are that (i) it decouples over products and (ii) it consequently involves only two variables (S_i and I_i^{Δ}). This makes it amenable to solution by a graphical approach. Note that, because the uncertain demand θ_i appears in the constraint set, the feasible region of problem IP_{SPS} varies for different demand realizations. This implies that, even though problem IP_{SPS} involves only two variables, it is not possible to a priori obtain a graphical representation of its feasible region. This problem can, however, be resolved by considering the *dual* of IP_{SPS}. In the resulting problem, because all cost coefficients are assumed to be deterministic, the feasible region will be independent of demand realizations. Thus, by associating nonnegative dual variables u_i, v_i, and w_i with constraints 7, 8, and 9, respectively, the dual of problem IP_{SPS} for each product i is given by

(DIP_i)

$$h_i A_i + \mu_i \theta_i + \max_{u_i, v_i, w_i} -\theta_i u_i - (A_i - I_i^L) v_i - A_i w_i$$

subject to

$$u_i + v_i + w_i \ge h_i + \mu_i - t_i \tag{13}$$

$$v_i \le \zeta_i \tag{14}$$

$$u_i, v_i, w_i \ge 0 \tag{15}$$

Under the assumption that the demand (θ_i) is always nonnegative, u_i can be eliminated from DIP_i using eq 13 to give the following equivalent formulation, after the constant term ($h_i A_i + \mu_i \theta_i$) is dropped.

(DIP'_i)

$$-(h_i + \mu_i - t_i)\theta_i + \max_{v_i, w_i} v_i(\theta_i - A_i + I_i^L) + w_i(\theta_i - A_i)$$

subject to

$$v_i + w_i \leq h_i + \mu_i - t_i \tag{16}$$

$$v_i \leq \zeta_i \tag{17}$$

$$v_i,\ w_i \geq 0 \tag{18}$$

For each product i, problem DIP$'_i$ involves only two variables v_i and w_i. A graphical representation of the feasible region can thus be obtained, as shown in Figure 1. Based on the feasible region illustrated in Figure 1, a cost assumption that has been implicitly incorporated is

$$h_i + \mu_i - t_i - \zeta_i \geq 0 \tag{19}$$

This assumption ensures economic feasibility of the production–distribution system. For an enterprise to sustain in the market, the revenue earned (μ_i) for a unit of finished product sold should be larger than the cumulative sum of the inventory holding (h_i), under-penalty (ζ_i), and transportation (t_i) costs.

The vector representing the gradient of the objective function, along with its corresponding v_i and w_i components, is also shown in Figure 1. The direction of this vector represents the direction of maximum ascent of the objective function value. The sign of the components of this vector determine which direction, out of those labeled I–IV in Figure 1, is the one of maximum ascent. It is not possible to determine this direction a priori because the two components depend on the realization of the stochastic demand. Therefore, four distinct cases should be considered for the four possible sign combinations of these two components. Based on this observation, the optimal solution for problem DIP$'_i$ can be characterized as follows:

Case I: $\theta_i - A_i \leq 0,\ \theta_i - A_i + I_i^L \leq 0$
$$\Rightarrow v_i = 0,\ w_i = 0$$

Case II: $\theta_i - A_i \leq 0,\ \theta_i - A_i + I_i^L \geq 0$
$$\Rightarrow v_i = \zeta_i,\ w_i = 0$$

Case III: $\theta_i - A_i \geq 0,\ \theta_i - A_i + I_i^L \geq 0$
$$\Rightarrow v_i = \zeta_i,\ w_i = h_i + \mu_i - t_i - \zeta_i$$

Case IV: $\theta_i - A_i \geq 0,\ \theta_i - A_i + I_i^L \leq 0$
$$\Rightarrow \text{Infeasible combination}$$

The resulting optimal objective function values for problem DIP$'_i$ are

Case I: $-(h_i + \mu_i - t_i)\theta_i$

Case II: $\zeta_i(I_i^L - A_i + \theta_i) - (h_i + \mu_i - t_i)\theta_i$

Case III: $\zeta_i I_i^L - (h_i + \mu_i - t_i)A_i$

Based on the dual optimal solutions obtained for problem DIP$'_i$, the corresponding primal optimal solutions for problem IP$_{SPS}$ can be reconstructed as

Case I: $I_i^\Delta = 0,\ S_i = \theta_i$

Case II: $I_i^\Delta = \theta_i - (A_i - I_i^L),\ S_i = \theta_i$

Case III: $I_i^\Delta = I_i^L,\ S_i = A_i$

Figure 2 shows the variation of the optimal supply policies with demand realization. As illustrated in the figure, the three distinct regions corresponding to the cases I, II, and III can be viewed as regimes of *low*-, *intermediate*-, and *high*-demand realizations, respectively. The transitions from low- to intermediate- and, subsequently, from intermediate- to high-demand regimes occur at demand realizations indicated by $\Theta_i^{L\to I}$ and $\Theta_i^{I\to H}$, respectively.

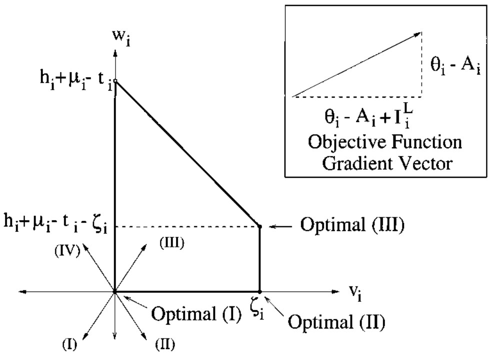

Figure 1. Feasible region and directions of steepest ascent of objective function for problem DIP$'_i$ (cases I–IV).

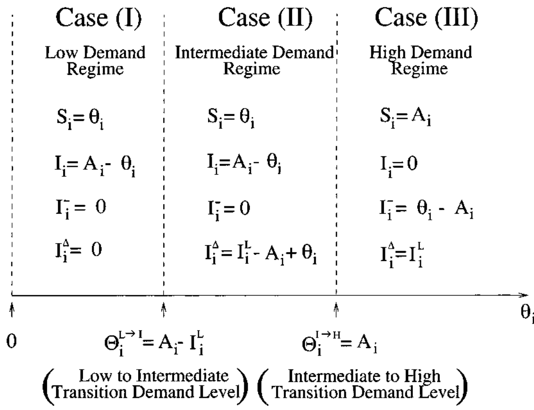

Figure 2. Optimal supply policies for the single-site case.

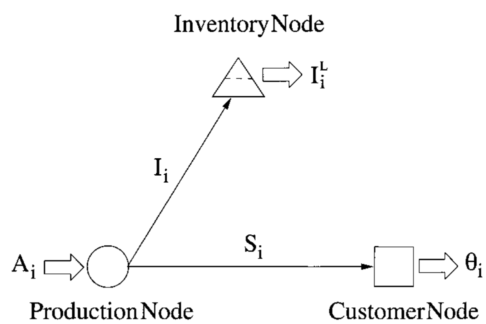

Figure 3. Network representation of problem IP$_{SPS}$.

An intuitive interpretation of the optimal supply policies shown in Figure 2 can be realized by considering the network representation of problem IP$_{SPS}$, as shown in Figure 3. Production variable A_i can be viewed as the amount of finished product available for supply at the production node prior to demand realization, as illustrated in Figure 3. This amount equals the sum of the initial inventory of the product and the total amount produced on all processors at the production site. Requirements for the product exists at two different demand nodes. There is an *internal* demand of I_i^L units at the inventory node and an *external* demand of θ_i units at the customer node, as shown in Figure 3. A key distinction between these two demands is that the internal demand is deterministic, whereas the external demand is uncertain. The flow from the production node to the inventory node corresponds to the inventory holding variable I_i. Similarly, the flow on the arc directed from the production node to the customer node corresponds to the customer supply variable S_i. Note that the inventory flow is allowed to exceed the internal demand (overstocking). However, overstocking at the customer is not permitted.[15]

Next, consider rewriting eq 19 as

$$-\mu_i + t_i \leq -\zeta_i + h_i$$

The left-hand side of the above expression represents the cost of shipping a unit of product from the production node to the customer node. Similarly, the right-hand side represents the net *minimum* cost incurred for the transfer of one product unit to the inventory node. This implies that, given a choice of satisfying only one of the two demands, the external demand should be given priority. This is referred to as the "customer priority principle".

In view of this, consider a specific realization of the external demand θ_i. If this realization is greater than the total supply A_i, then based on the customer priority principle, everything must be shipped to the customer. The inventory level will drop to zero, and thus, the maximum possible underpenalty cost corresponding to a deviation of I_i^L units will be incurred. This supply policy corresponds exactly to the high-demand regime given by case III. In this case, neither of the two demands is completely satisfied. Now consider the other extreme case in which the supply is large enough to completely satisfy both demands, i.e., $A_i \geq I_i^L + \theta_i$. The optimal supply policy in this case would involve shipping θ_i units to the customer and holding the remaining $A_i - \theta_i$ units in inventory. This would be equivalent to case I. For the intermediate-demand regime corresponding to case II in Figure 2, the entire external demand can be met only at the expense of incurring some underpenalty cost. In this case, after shipping θ_i units to the customer, the remaining $A_i - \theta_i$ units are transferred to inventory. This results in a deviation of $I_i^L - (A_i - \theta_i)$ units below the target safety stock level.

The basic idea of obtaining an explicit solution of the inner inventory-management problem is similar in spirit to the parametric programming approach of Acevedo and Pistikopoulos.[33] The three demand regimes identified are equivalent to the *critical regions* in the parametric programming framework in which different bases are optimal. Similarly, the transition demand levels correspond to the *critical points* at which the change of optimal bases occurs.[34] The work of Pistikopoulos and co-workers uses sampling based numerical integration techniques for expectation evaluation. In contrast, in this work an analytical method for calculating the expectation using the explicitly derived optimal solution is utilized, as described next.

Expectation Evaluation. The calculation of the expected value of the solution of the inner problem requires integration over all possible demand realizations. To facilitate this calculation, define α_i^L, α_i^I, and α_i^H as the probabilities that measure the likelihood of the demand of a particular product being low, intermediate, and high, respectively. Therefore

$$\alpha_i^L = \Pr[\theta_i \leq A_i - I_i^L] \tag{20}$$

$$\alpha_i^I = \Pr[A_i - I_i^L \leq \theta_i \leq A_i] \tag{21}$$

$$\alpha_i^H = \Pr[\theta_i \geq A_i] \tag{22}$$

Application of the probability-scaled additive property of the expectation operator to the inner problem optimal value yields the following recourse function $Q_i(A_i)$.

$$Q_i(A_i) = E_{\theta_i}\begin{bmatrix} h_iA_i + \mu_i\theta_i \\ + [\,-(h_i + \mu_i - t_i)\theta_i \mid \theta_i \leq A_i - I_i^L] \\ + [\zeta_i(I_i^L - A_i + \theta_i) - (h_i + \mu_i - t_i)\theta_i \mid A_i - I_i^L\ \theta_i \leq A_i\,] \\ + [\zeta_iI_i^L - (h_i + \mu_i - t_i)A_i \mid A_i\ \theta_i] \end{bmatrix}$$

$$\begin{aligned} &= h_iA_i + \mu_i\theta_i^m \\ &\quad + \alpha_i^L E_{\theta_i}[-(h_i + \mu_i - t_i)\theta_i \mid \theta_i \leq A_i - I_i^L] \\ &\quad + \alpha_i^I E_{\theta_i}[\zeta_i(I_i^L - A_i + \theta_i) - (h_i + \mu_i - t_i)\theta_i \\ &\qquad\qquad \mid A_i - I_i^L\ \theta_i \leq A_i] \\ &\quad + \alpha_i^H E_{\theta_i}[\zeta_iI_i^L - (h_i + \mu_i - t_i)A_i \mid A_i \leq \theta_i] \end{aligned} \tag{23}$$

where θ_i^m is the mean demand for product i. The analytical evaluation of the integrals involved in eq 23 is facilitated by standardizing the demand as

$$z_i = \frac{\theta_i - \theta_i^m}{\sigma_i} \tag{24}$$

and defining

$$K_i^1 = \frac{A_i - I_i^L - \theta_i^m}{\sigma_i} \text{ and } K_i^2 = \frac{A_i - \theta_i^m}{\sigma_i} \tag{25}$$

where σ_i is the standard deviation of the demand for product i. The probabilities for low-, intermediate-, and high-demand realizations are then calculated as

$$\alpha_i^L = \Pr[z_i \leq K_i^1] = \Phi(K_i^1) \tag{26}$$

$$\alpha_i^I = \Pr[K_i^1 \leq z_i \leq K_i^2] = \Phi(K_i^2) - \Phi(K_i^1) \tag{27}$$

$$\alpha_i^H = \Pr[K_i^2 \leq z_i] = 1 - \Phi(K_i^2) \tag{28}$$

where $\Phi(\cdot)$ denotes the standardized normal cumulative distribution function. Application of the definition of expectation for a normally distributed random variable yields the following conditional expectations:

$$E[z_i \mid z_i \le K_i^1] = \frac{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{K_i^1} z_i e^{-1/2 z_i^2}\, dz_i}{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{K_i^1} e^{-1/2 z_i^2}\, dz_i} = -\frac{f(K_i^1)}{\Phi(K_i^1)} \quad (29)$$

$$E[z_i \mid K_i^1 \le z_i \le K_i^2] = \frac{\frac{1}{\sqrt{2\pi}} \int_{K_i^1}^{K_i^2} z_i e^{-1/2 z_i^2}\, dz_i}{\frac{1}{\sqrt{2\pi}} \int_{K_i^1}^{K_i^2} e^{-1/2 z_i^2}\, dz_i} = -\frac{f(K_i^2) - f(K_i^1)}{\Phi(K_i^2) - \Phi(K_i^1)} \quad (30)$$

where $f(\cdot)$ is the normal density function. Incorporation of the expressions for the conditional expectations and the probabilities in eq 23 yields

$$Q_i(A_i) = \begin{bmatrix} \sigma_i \zeta_i\, [K_i^1 \Phi(K_i^1) + f(K_i^1)] \\ + \sigma_i (h_i + \mu_i - t_i - \zeta_i)\, [K_i^2 \Phi(K_i^2) + f(K_i^2)] \\ - \mu_i \sigma_i K_i^2 + \zeta_i I_i^L + t_i A_i \end{bmatrix} \quad (31)$$

Having calculated the expectation of the optimal value of the inner problem, the original two-stage formulation 2SMP is recast as the following single-stage deterministic equivalent problem.

(DEQ_{SPS})

$$\min_{\substack{P_{ij}, RL_{ij}, FRL_{fj} \\ C_i, A_i, Y_{fj}}} \sum_{f,j} FC_f Y_{fj} + \sum_{i,j} v_{ij} P_{ij} + \sum_i p_i C_i + \sum_i Q_i(A_i)$$

subject to

eqs 1−6, 11, and 25

Problem DEQ_{SPS} is a mixed-integer, nonlinear problem (MINLP). The nonlinear terms, which are restricted to the objective function, are of the general form

$$g(K) = K\Phi(K) + f(K) \quad (32)$$

The convexity of $g(K)$ is readily established by recognizing that its second derivative is always nonnegative.[3] Therefore, problem DEQ_{SPS} is a convex MINLP and can be solved to global optimality by techniques such as generalized benders decomposition (GBD)[35] or outer approximation (OA).[36]

The convexity of problem DEQ_{SPS} is a consequence of the *complete recourse* property of the inner inventory-management problem. A feasible second-stage solution exists for *any* demand realization and *any* production setting, as the supply policy of not shipping anything to the customer and transferring the entire production amount to inventory is always feasible. The feasible uncertainty region consisting of all possible demand realizations is, therefore, *independent* of the first-stage decisions. This special problem structure leads to the convexity of the proposed formulation. Alternately, the convexity property can also be inferred within the parametric programming framework by recognizing that the first-stage variable appears only as the *right-hand side* vector in the second-stage problem.

Multiple Production Sites

Having addressed the simpler single-site version of the problem 2SMP, next consider the general multisite case. The inner optimization problem for this case is given by IP_{MPS}.

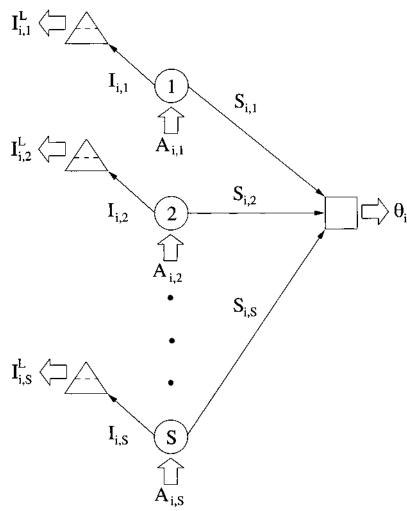

Figure 4. Network representation of problem IP_{MPS}.

(IP_{MPS})

$$\sum_{i,s} h_{is} A_{is} + \sum_i \mu_i \theta_i + \min_{I_{is}^{\Delta}, S_{is}} \sum_{i,s} \zeta_{is} I_{is}^{\Delta} - \sum_{i,s} (h_{is} + \mu_i - t_{is}) S_{is}$$

subject to

$$\sum_s S_{is} \le \theta_i \quad (33)$$

$$S_{is} - I_{is}^{\Delta} \le A_{is} - I_{is}^L \quad (34)$$

$$S_{is} \le A_{is} \quad (35)$$

$$S_{is}, I_{is}^{\Delta} \ge 0 \quad (36)$$

where

$$A_{is} = I_{is}^0 + \sum_j P_{ijs} - \sum_{s'} W_{iss'} \quad (37)$$

Comparison of problem IP_{MPS} with the single-site inner optimization problem IP_{SPS} indicates that the presence of multiple production facilities is reflected in eq 33, which allocates product supply from different sites to meet the customer demand. This constraint couples the production sites by enforcing no overstocking at the customer. In addition, the transfer of intermediate products between production sites is accounted for in the definition of A_{is} by eq 37.

The separability over the set of products observed for the single-site case is also preserved for the multisite case. The network representation of IP_{MPS} for a given product is shown in Figure 4. The demand at the customer can now be met by multiple manufacturing facilities. Because the total number of variables involved in IP_{MPS} is equal to twice the number of production sites, it is not amenable to direct graphical solution. Therefore, a different solution strategy, utilizing the insight gained from the single-site case, is employed. First, a *primal* feasible solution for problem IP_{MPS} is postulated by extending the results obtained for problem IP_{SPS}. Subsequently, a *dual* feasible solution having the same objective function value as the postulated primal solu-

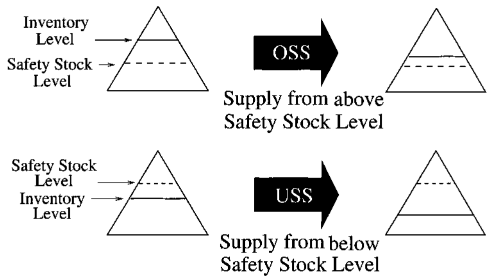

Figure 5. Schematic illustration of γ_s (OSS) and ω_s (USS).

tion is constructed. This establishes the optimality of the postulated primal solution based on the *strong duality* theorem of linear programming.

The construction of the primal feasible solution for IP_{MPS} requires the introduction of additional notation and assumptions. The key questions that need to be answered for the multisite case are as follows: (1) Which sites service the customer in the three demand regimes? (2) What is the relative supply ranking in the three demand regimes? To answer the first question, based on the relative magnitudes of A_s and I_s^L (the index i is dropped in all further analysis for clarity of presentation), the sites are classified as

$$IS = \{s \in S \mid A_s - I_s^L \geq 0\}$$

$$ID\ \{s \in S \mid A_s - I_s^L < 0\}$$

with $IS \cup ID = S$ and $IS \cap ID = \phi$. Sites belonging to set IS are referred to as *internally sufficient* (IS) sites while those constituting set ID are termed *internally deficient* (ID). This terminology stems from the interpretation of $A_s - I_s^L$ as the production amount available in excess of the internal demand at a particular production site. Note that different IS and ID site classification sets exist for each product i.

To answer the second question, two additional cost parameters need to be defined as

$$\gamma_s = t_s - h_s \tag{38}$$

$$\omega_s = t_s - h_s + \zeta_s \tag{39}$$

A schematic description of γ_s and ω_s, which are defined as the *over-safety stock-supply* cost (OSS) and the *under-safety stock-supply* cost (USS), respectively, is given in Figure 5. OSS represents the cost incurred after a unit of product is transferred from the inventory to the customer (not including the revenue earned) from *above* the safety stock level. Alternately, if the product is supplied from *below* the safety stock level, the cost incurred is USS. These two cost parameters play an important role in determining the order in which the sites service the customer in a particular demand regime.

The following cost assumptions are also enforced.

$$\mu + h_s - t_s - \zeta_s \geq 0 \tag{40}$$

$$t_s \leq h_s \leq \zeta_s \tag{41}$$

Equation 40 is simply an extension of the customer priority principle to the multisite setting. The assumption that ζ_s is greater than h_s is consistent with maintaining an inventory target of I_s^L. The assumption that inventory holding cost h_s exceeds the transportation

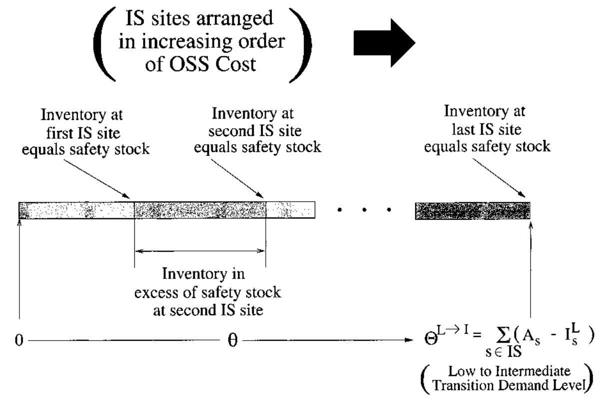

Figure 6. Low-demand regime for the multisite case.

cost t_s typically holds for most local and global supply chains. In the latter, because of appreciable transit lead times, the transportation charges are usually viewed as surrogates for inventory holding costs.

An important relationship that results from eq 41 is

$$\gamma_s \leq 0 \leq \omega_s \tag{42}$$

Equation 42 implies that, given a choice for shipping a unit of product from above or below the safety stock level, the former should be chosen for minimizing cost. This key result is exploited next for postulating optimal supply policies in the low-, intermediate-, and high-demand regimes.

Low-Demand Regime. For the single-site case, the low-demand regime consists of all possible demand realizations for which *zero* underpenalty charges are incurred. In the multisite setting, however, because of the presence of the ID sites, underpenalty charges may be unavoidable. Therefore, for the multisite case, the low-demand regime is defined as the set of all possible demand realizations for which *minimum total* underpenalty charges are incurred in the supply chain. This corresponds to limiting the underpenalty charges to only the ID sites and not violating the safety stock levels at any of the IS sites.

If the external demand is zero, the available amount A_s is transferred to inventory at all sites, as no overstocking is permitted at the customer. Subsequently, any nonzero demand can be met at minimum cost by shipping the product from the IS site with the lowest OSS cost. The maximum demand that can be allocated to this site before switching to the site with the second lowest OSS cost is the inventory in excess of the target safety stock level (i.e., $A_s - I_s^L$). This prevents the inventory level at the first (lowest OSS cost) site from dipping below the target safety stock level avoiding underpenalty charges.

Based on these observations, feasible supply policies for the IS sites can be obtained by ranking them in increasing order of OSS cost (Figure 6). The resulting sequence of sites, starting with the site with the lowest OSS cost, represents the order in which the IS sites service the customer. In accordance with the definition of the low-demand regime, $A_s - I_s^L$ units are shipped from a given site before switching to the next one to maintain the underpenalty charges at a minimum. The entire available amount at the ID sites is transferred to inventory to minimize underpenalty charges.

The production sites are reordered to obtain a concise mathematical representation of the above-described supply policies such that the condition

$$\gamma_{s-1} \leq \gamma_s \tag{43}$$

holds. Also, define site $s_l^* \in IS$ as

$$\sum_{\substack{s=1 \\ s\in IS}}^{s_l^*-1} (A_s - I_s^L) \leq \theta \leq \sum_{\substack{s=1 \\ s\in IS}}^{s_l^*} (A_s - I_s^L) \tag{44}$$

Note that the summations in eq 44 consider only the IS sites. The supply policies for the different sites are then given by

$$\left.\begin{aligned} S_s &= 0 \\ I_s &= A_s \\ I_s^\Delta &= I_s^L - A_s \end{aligned}\right\} \forall\ s \in ID \tag{45}$$

$$\left.\begin{aligned} S_s &= A_s - I_s^L \\ I_s &= I_s^L \\ I_s^\Delta &= 0 \end{aligned}\right\} \forall\ s \leq s_l^* - 1,\ s \in IS \tag{46}$$

$$\left.\begin{aligned} S_s &= \theta - \sum_{\substack{s_1=1 \\ s\in IS}}^{s_l^*-1} (A_s - I_s^L) \\ I_s &= A_s - \theta + \sum_{\substack{s=1 \\ s\in IS}}^{s_l^*-1} (A_s - I_s^L) \\ I_s^\Delta &= 0 \end{aligned}\right\} s = s_l^*,\ s \in IS \tag{47}$$

$$\left.\begin{aligned} S_s &= 0 \\ I_s &= A_s \\ I_s^\Delta &= 0 \end{aligned}\right\} \forall\ s \geq s_l^* + 1,\ s \in IS \tag{48}$$

For a low-demand realization given by eq 44, eqs 45–48 summarize the supply policies obtained by sequentially allocating demand to the IS sites. Ranking of the IS sites on the basis of the OSS cost is achieved through eq 43, and no shortage at the customer ($I^- = 0$) exists in the low-demand regime. The transition demand level $\Theta^{L\to I}$ is thus given by

$$\Theta^{L\to I} = \sum_{s\in IS} (A_s - I_s^L)$$

as shown in Figure 6.

Intermediate-Demand Regime. Demand realizations exceeding $\Theta^{L\to I}$ comprise the intermediate-demand regime. For a demand realization of $\Theta^{L\to I}$, based on the analysis presented for the low-demand regime, the supply policies consist of shipping $A_s - I_s^L$ units from *each* of the IS sites. As a result, the inventory levels at all of the IS sites are driven to I_s^L. Thus, supply of an additional unit of product to the customer causes the inventory level of one of the sites to fall below the target safety stock level. This implies that, irrespective of the type of site (i.e., IS or ID) chosen to ship this additional unit, a USS cost is incurred. Therefore, to minimize cost, the candidate site for supplying this additional unit (over and above $\Theta^{L\to I}$) to the customer is the one with the lowest USS cost.

The above observation suggests that a procedure based on ranking the sites with respect to the USS cost can be used to allocate the portion of demand in excess of $\Theta^{L\to I}$. Thus, for an intermediate demand $\theta \geq \Theta^{L\to I}$, the allocation of demand can be achieved in two phases. In the first phase, $\Theta^{L\to I}$ units are allocated to the IS sites based on their OSS cost rank. In the second phase, the remaining $\theta - \Theta^{L\to I}$ units are allocated by re-ranking *all* of the sites in terms of their USS cost (Figure 7).

In the spirit of the low-demand regime, ranking of the sites on the basis of their USS cost is achieved by reordering the production sites such that the condition

$$\omega_{s-1} \leq \omega_s \tag{50}$$

holds. Let $s_i^* \in IS \cup ID$ be the site for which

$$\Theta^{L\to I} + \sum_{\substack{s=1 \\ s\in ID}}^{s_i^*-1} A_s + \sum_{\substack{s=1 \\ s\in IS}}^{s_i^*-1} I_s^L \leq \theta \leq \Theta^{L\to I} + \sum_{\substack{s=1 \\ s\in ID}}^{s_i^*} A_s + \sum_{\substack{s=1 \\ s\in IS}}^{s_i^*} I_s^L \tag{51}$$

For this intermediate-demand realization, the supply policies for the ID and IS sites are given by

$$\left.\begin{aligned} S_s &= A_s \\ I_s &= 0 \\ I_s^\Delta &= I_s^L \end{aligned}\right\} \forall\ s \leq s_i^* - 1 \tag{52}$$

$$\left.\begin{aligned} S_s &= \theta - \sum_{s=1}^{s_i^*-1} A_s - \sum_{\substack{s\geq s_i^*+1 \\ s\in IS}} (A_s - I_s^L) \\ I_s &= A_s - \theta + \sum_{s=1}^{s_i^*-1} A_s + \sum_{\substack{s\geq s_i^*+1 \\ s\in IS}} (A_s - I_s^L) \\ I_s^\Delta &= I_s^L - A_s + \theta - \sum_{s=1}^{s_i^*-1} A_s - \sum_{\substack{s\geq s_i^*+1 \\ s\in IS}} (A_s - I_s^L) \end{aligned}\right\} s = s_i^* \tag{53}$$

$$\left.\begin{aligned} S_s &= 0 \\ I_s &= A_s \\ I_s^\Delta &= I_s^L - A_s \end{aligned}\right\} \forall\ s \geq s_i^* + 1,\ s \in ID \tag{54}$$

$$\left.\begin{aligned} S_s &= A_s - I_s^L \\ I_s &= I_s^L \\ I_s^\Delta &= 0 \end{aligned}\right\} \forall\ s \geq s_i^* + 1,\ s \in IS \tag{55}$$

Equations 52–55 represent the supply policies for an intermediate-demand realization given by eq 51. Consequently, the transition demand level $\Theta^{I\to H}$ is given by

$$\Theta^{I\to H} = \sum_s A_s \tag{56}$$

as shown in Figure 7. As in the low-demand regime, there is no shortage at the customer ($I^- = 0$).

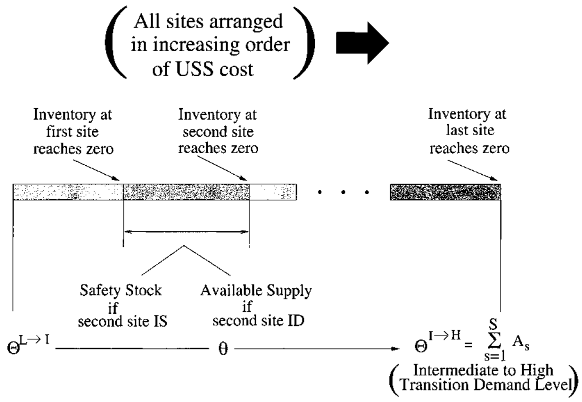

Figure 7. Intermediate-demand regime for the multisite case.

High-Demand Regime. In the low- and intermediate-demand regimes, the entire customer demand is met, and no sales are lost (i.e., $I^- = 0$). For high-demand realizations (i.e., $\theta \geq \Theta^{I\to H}$), however, this is no longer possible. Therefore, based on the customer priority paradigm, the entire production amount from all of the sites is transferred to the customer location resulting in the following optimal supply policies.

$$\left.\begin{array}{l} S_s = A_s \\ I_s = 0 \\ I_s^{\Delta} = I_s^L \\ I^- = \theta - \sum\limits_{s \in IS \cup ID} A_s \end{array}\right\} \forall\, s \in IS \cup ID \tag{57}$$

Complete inventory depletion at all of the manufacturing facilities results in maximum underpenalty charges in the supply chain.

Next, the feasibility and optimality of the postulated supply policies are ascertained. The methodology adopted for allocating demand to the various production sites ensures that the amount supplied from a site does not exceed the total amount available for supply (A_s). This guarantees nonnegative inventory levels at the sites. The no-overstocking restriction is also enforced by not shipping in excess of the demand in the low- and intermediate-demand regimes, thereby ensuring the feasibility of the constructed primal solution. The optimality of this solution is established based on LP duality (see Appendix A for the proof). Subsequently, as for the single-site case, the expectation evaluation is carried out analytically. The derived expressions are given in Appendix B.

Deterministic Equivalent Formulation

The derivation of the deterministic equivalent formulation of the multisite case requires the classification of sites into types ID and IS. To this end, the disjunctive programming approach of Balas[37] is utilized. This involves associating binary variables δ_s^+ and δ_s^- defined as

$$\delta_s^+ = \begin{cases} 1 \text{ if } s \in IS \\ 0 \text{ otherwise} \end{cases} \text{and } \delta_s^- = \begin{cases} 1 \text{ if } s \in ID \\ 0 \text{ otherwise} \end{cases} \tag{58}$$

with each of the production sites in the supply chain. Because IS and ID are disjoint sets, the condition

$$\delta_s^+ + \delta_s^- = 1 \tag{59}$$

must hold. The disjunction is extended to the production variable A_s by defining

$$A_s = A_s^+ + A_s^- \tag{60}$$

in conjunction with the restrictions

$$I_s^L \delta_s^+ \leq A_s^+ \leq A_s^{UP} \delta_s^+ \tag{61}$$

$$0 \leq A_s^- \leq I_s^L \delta_s^- \tag{62}$$

where A_s^{UP} denotes an upper bound on A_s and is given by

$$A_s^{UP} = I_s^0 + \sum_j R_{js} H_{js} \tag{63}$$

based on the first-stage production setting constraints. Under this transformation, the standardized variables $K^1_{s_l^*}$ and $K^2_{s_l^*}$ are given by

$$\sigma K^1_{s_l^*} = [\sum_{s=1}^{s_l^*} (A_s^+ - I_s^L \delta_s^+) - \theta^m] \tag{64}$$

$$\sigma K_{s_i^*} = [\Theta^{L\to I} + \sum_{s=1}^{s_i^*} (A_s^- + I_s^L \delta_s^+) - \theta^m] \tag{65}$$

Consequently, after the expected second-stage costs are included (eq 74 in Appendix B), the deterministic equivalent formulation for the multisite case is given by

$(\mathrm{DEQ_{MPS}})$

$$\min_{\substack{P_{ijs}, RL_{ijs}, FRL_{fjs} \\ C_{is}, Y_{fjs}, A_{is}, W_{iss'}}} \sum_{f,j,s} FC_f Y_{fjs} + \sum_{i,j,s} v_{ijs} P_{ijs} + \sum_{i,s} p_{is} C_{is} + \sum_{i,s,s'} t_{iss'} W_{iss'} + \sum_i Q_i(A_{is})$$

subject to

$$\text{eqs } 1-6, 37, 58-65, \text{ and } 74$$

Note that the product index i has been reintroduced in the formulation.

Illustrative Example

The proposed methodology is first highlighted with a small three-site supply-chain example. A single product is produced at each one of these sites on a single processor. The parameters characterizing this supply chain are listed in Table 1. The initial inventory at all three sites is zero, and the revenue earned per unit product μ is equal to 5.0. The uncertain product demand θ is given by $N(110,30)$.

The deterministic single-stage equivalent problem $\mathrm{DEQ_{MPS}}$ is solved for the example supply chain using OA. The optimal objective value obtained is 291 with

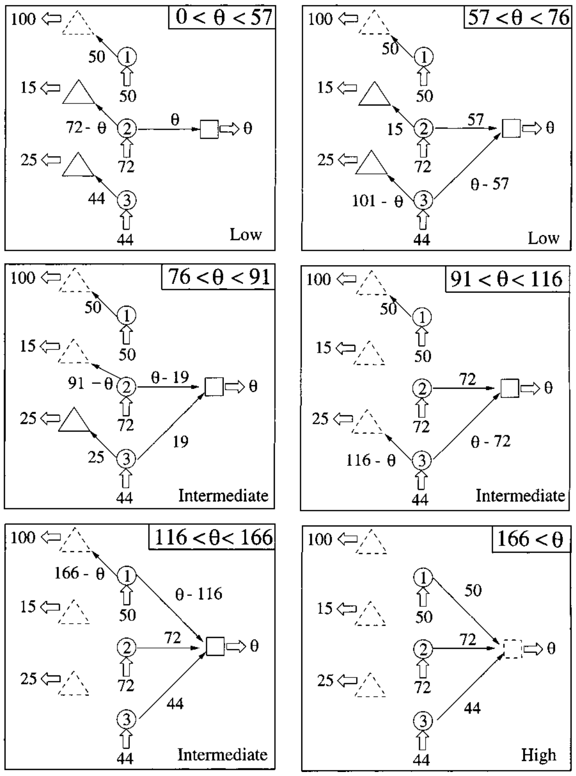

Figure 8. Supply policies for motivating example.

Table 1. Parameters for the Motivating Example

s	FC_s	v_s	R_s	H_s	MRL_s	t_s	h_s	ζ_s	I_s^L	γ_s	ω_s
1	4.5	0.5	0.5	100	50	0.1	0.8	1.7	100	−0.7	1.0
2	6.5	0.3	0.6	120	25	0.2	0.7	1.3	15	−0.5	0.8
3	5.0	0.6	0.5	150	20	0.3	0.6	1.2	25	−0.3	0.9

Table 2. Optimal First-stage Production Policies for the Motivating Example

s	Y_s	RL_s	P_s
1	1	100	50
2	1	120	72
3	1	88	44

the optimal first-stage planning decisions given in Table 2. The optimal production run lengths RL_s and amounts P_s are listed in Table 2. The detailed supply policies for the three production sites, shown in Figure 8, illustrate the flow on the supply and inventory arcs for all demand regimes. The safety stock levels I_s^L imply that site 1 is of type ID, while sites 2 and 3 are of type IS. Therefore $ID = \{1\}$ and $IS = \{2,3\}$. The ranking of the IS sites on the basis of their OSS costs determines the supply policies in the low-demand regime. The first 57 product demand units are allocated to site 2, which is the IS site with the lowest OSS cost. Product demand between 57 and 76 units is assigned to site 3, as shown in Figure 8. This demand level of 76 units corresponds to the transition demand level $\Theta^{L\rightarrow I}$. By not allowing the inventory levels to fall below the safety stock level at sites 2 and 3, the underpenalty charges are restricted to only site 1 (where they are unavoidable). The intermediate-demand regime is composed of product demand orders between 76 and 166 units. Ranking of the three sites with respect to their USS costs determines the optimal supply policy in the intermediate regime. Site 2 is the site with the lowest USS cost. Demand is initially allocated to site 2 until the inventory at site 2 is completely depleted. At this point, supply is redirected to the site with the second lowest USS cost (site 3). Upon depletion of inventory in site 3, product demand is allocated to site 1, as shown in Figure 8. This determines the transition demand level $\Theta^{I\rightarrow H}$. The supply policies in the high-demand regime consist of shipping the entire production amount at the three sites to the customer. By parametrically solving the inner optimization problem, the optimal supply policies for the entire range of demand realizations are identified. Given *any* demand realization, the supply policy that minimizes the expected cost can thus be ascertained.

Table 3. Variation of EEV, RP, and VSS with Demand Standard Deviation for the Illustrative Example

σ	EEV	RP	VSS	$\frac{VSS}{RP} \times 100$
10	287	285	2	0.70
15	292	286	6	2.10
20	298	287	11	3.83
25	305	288	17	5.90
30	313	291	22	7.56
35	321	294	27	9.18

In light of these results, it is important to quantify the impact of uncertainty on the planning decisions. This can be accomplished on the basis of the *value of the stochastic solution* (VSS),[26] which evaluates the cost of ignoring uncertainty. By replacing all random parameters by their expected values and solving the resulting deterministic *expected-value* (EV) problem, the *EV solution* is obtained. Subsequently, the recourse problem (RP) is solved with the first-stage decisions fixed at the EV solution. The resulting optimal value is known as the *expected result of using the EV solution* (EEV).[26] EEV measures how the EV solution performs in the face of uncertainty. The VSS is then defined by

$$\mathrm{VSS} = \mathrm{EEV} - \mathrm{RP}$$

Table 3 lists the EEV and the VSS values obtained for the illustrative example as the standard deviation of the demand is varied. The monotonically increasing values of the VSS indicate that the cost of neglecting uncertainty increases with the degree of uncertainty. Under high-risk conditions, savings of approximately 9% are achieved, justifying the inclusion of uncertainty in the planning decisions. In the following example, a computational comparison between the discretization methods and the proposed solution technique is provided for a larger supply chain.

Example 1

The first example, initially proposed by McDonald and Karimi,[15] consists of 34 products that are manufactured at two consecutive production sites. The first site produces 23 products grouped into 11 product families. Some of these 23 products are shipped as intermediates to the second site, which produces the remaining 11 products. Demands for all products are present at the beginning of each one of the 12 time periods of 1 month duration. A detailed description of the problem can be found in McDonald and Karimi.[15]

First, the single-period version of this problem instance is considered. The product demands are assumed to be normally distributed with a standard deviation of 20% of the expected demand. The resulting problem DEQ_{MPS} is solved by a customized implementation of

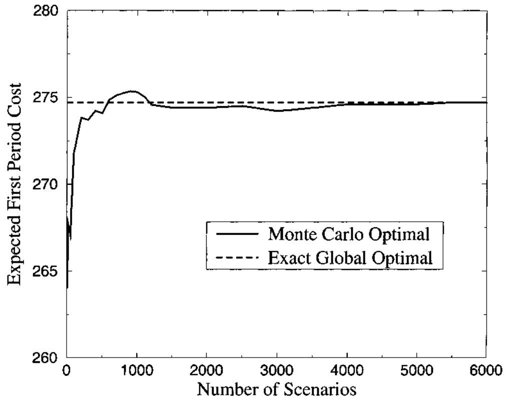

Figure 9. Comparison of a Monte Carlo sampling implementation with the proposed solution procedure.

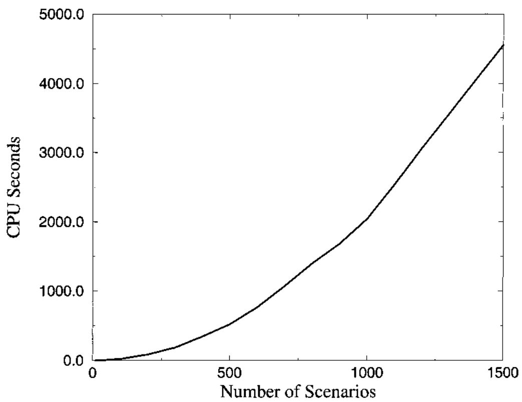

Figure 10. Variation of computational time with number of scenarios sampled.

the OA algorithm.[36] The lower and upper bounds obtained are 274.70 and 274.74, respectively. These are obtained in only 5 iterations of the OA implementation utilizing a total of 2 CPU s. For comparison, the single-period problem is also solved using Monte Carlo sampling.[26] This involves the generation of a large number of demand scenarios and the incorporation of supply-chain variables for each one of these scenarios in the inner optimization problem, yielding an MILP formulation. This MILP is solved for an increasing number of scenarios, and the results are shown in Figure 9. As expected, the Monte Carlo optimal value approaches the exact optimal objective function as the number of considered scenarios increases. However, over 1000 of them are needed for good agreement. Figure 10 shows the computational resources expended in obtaining these results, which scale exponentially with the number of scenarios, in accordance with the NP-hard nature of the MILP problem. Computational savings of almost 2 orders of magnitude over Monte Carlo sampling (2 CPU s as compared to 1067 CPU s for 700 scenarios) clearly highlight the benefits of the proposed methodology.

Next is examined what quantitative benefit, if any, is achieved by incorporating a description of uncertainty for a multiperiod planning framework. To answer this question, the following multiperiod simulation study is conducted. Consider two planners: a stochastic planner (S), who has information about both the mean and the standard deviation of the demand; and a deterministic planner (D), who has information only about the mean demand. Both of these planners plan on a rolling horizon basis, as shown in Figure 11. In the first period, planner S solves the stochastic formulation, while planner D

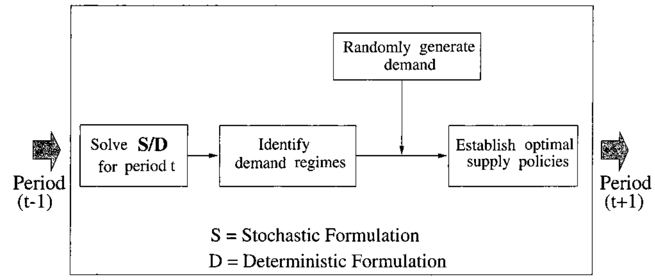

Figure 11. Simulation procedure adopted for the multiperiod setting.

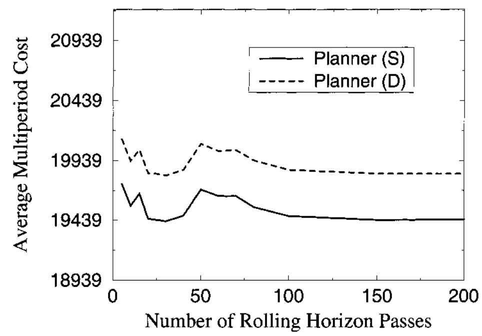

Figure 12. Multiperiod simulation results for the stochastic and the deterministic planners.

solves the deterministic formulation. This results in two alternative optimal production policies for the first period. Based on the optimal values taken by the production variables, the two planners identify the three demand regimes for each product. Subsequently, randomly generated demand realizations are revealed to both planners (Figure 11). Based on whether the demand realized is low, intermediate, or high, the supply policies for each product are determined by the two planners, along with the *actual* second-stage costs. The optimal supply policies define the initial conditions for the second period. For example, the inventory level as determined by the supply policies defines the initial inventory for the second period. Similarly, the shortage at the customer at the end of the first period is incorporated into the mean demand for the second period. This procedure is carried out in a rolling horizon manner for the 12-month planning period. It is repeated a number of times to average over the randomly generated demands revealed to the two planners.

The performance of the two planners is shown in Figure 12, where the running average optimal expected costs for the two planners are plotted against an increasing number of demand randomizations. Clearly, the stochastic planner consistently outperforms the deterministic planner by identifying better planning policies. In the limit, expected values of the multiperiod costs obtained reach 19643 and 20027 for planner S and D, respectively. These represent cost savings of approximately 2% by the stochastic planner. This difference in the expected costs can be interpreted as the savings achieved solely by including a description of demand variability, quantified in terms of the standard deviation of the uncertain demand, into the planning process.

Example 2

The second example problem consists of a larger supply chain involving six production sites manufactur-

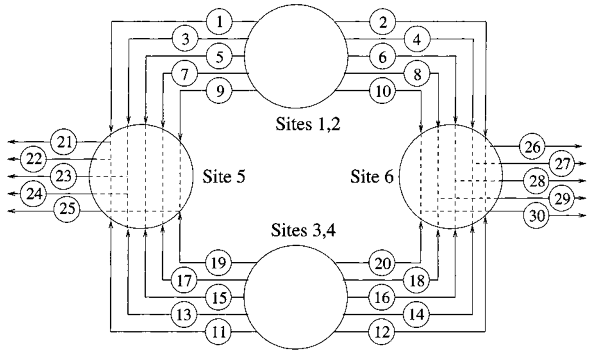

Figure 13. Supply chain for second example.

Table 4. Monte Carlo (MC) Results for Example 2

# scenarios	MC optimal	CPU
10	1441	2108
50	1497[a]	≥10000
100	1544[b]	≥10000

[a] 2% optimality gap. [b] 4% optimality gap.

ing a total of 30 products, as illustrated in Figure 13. Sites 1 and 2 (3 and 4) produce the same products, 1−10 (11−20). However, these sites are characterized by different production characteristics and cost parameters. These products are either shipped as finished products to the customer or as intermediate products to sites 5 and 6, as shown in Figure 13. An assembly-type product structure exists at sites 5 and 6, where each finished product is produced from two intermediate products. All sites consist of a single processor that is capacity constrained, and fixed setup charges are incurred at each site.

To assess the computational complexity of the problem, the deterministic version of the problem is solved first. The optimal deterministic plan incurs a total cost of 1332.54 obtained after 145 CPU s. Subsequently, the stochastic problem is solved with the customized OA algorithm. This identifies lower and upper bounds of 1509.14 and 1509.62 respectively in 9 iterations of the algorithm and 2372 s of CPU time. The increased objective value over the deterministic optimal objective value reflects the cost of uncertainty at the planning stage. The same problem instance is also solved using Monte Carlo sampling. The results obtained (see Table 4) illustrate the widening gap between analytical integration and stochastic sampling for larger problem instances.

Summary and Conclusions

In this paper, a two-stage modeling and solution framework was proposed for incorporating demand uncertainty in midterm planning problems. The midterm planning model of McDonald and Karimi was adopted as the reference model. Specifically, the supply chains considered were characterized by (semi)continuous processes and multiple production sites. The partitioning of the variables and constraints of the model into production and supply chain provided the appropriate structure for a two-stage stochastic programming formulation. The production decisions, because of their appreciable lead times, were made in a here-and-now fashion before the uncertainty in demand was resolved. Subsequently, the wait-and-see supply-chain decisions were made on the basis of the production decisions and the realization of the demand.

The expectation evaluation of the inner recourse problem was resolved in two steps. The first step involved obtaining a closed-form solution of the inner problem using LP duality. This analysis led to three different optimal supply policies depending on whether the product demand was within the low-, intermediate-, or high-demand regime. Based on the insight obtained from the single-site case, the multisite case was subsequently resolved. Two key issues identified in the analysis for the multisite case were (i) the classification of sites into types IS and ID and (ii) the ranking of sites on the basis of the OSS and USS costs in the low- and intermediate-demand regimes. The second step was the computation of the expectation of the second-stage costs by analytical integration. The resulting single-stage deterministic equivalent MINLP was shown to have a convex continuous part. A customized version of OA[36] was implemented. Computational results for multisite problems indicated that the proposed analysis and solution framework was at least an order of magnitude more efficient than sampling methods such as Monte Carlo integration at the expense of restricting the modeling of uncertainty to normal. In addition, a comparitive study between the planning suggestions of a deterministic model and the proposed two-stage stochastic model showed that planning savings can be realized by recognizing and incorporating demand uncertainty in the decision-making framework.

It is important to note that the normality assumption for the uncertain demands plays a key role in the expectation evaluation step of the proposed methodology. Extension of this work to account for a general probability distribution is under consideration. Incorporation of uncertainty in the second-stage cost parameters such as revenue, transportation cost, and underpenalty cost within the proposed analytical framework is also being explored. Furthermore, application of the proposed methodology to the more general multiperiod and multicustomer problem is under investigation.

Acknowledgment

We acknowledge useful discussions with Dr. Conor McDonald and financial support by NSF-GOALI Grant CTS-9907123 and DuPont Educational Aid Grant 1999/2000.

Notation

Sets

$I = \{i\}$ = set of products

$I^{RM} \subset I = \{i\}$ = set of raw materials

$I^{IP} \subset I = \{i\}$ = set of intermediate products

$I^{FP} \subset I = \{i\}$ = set of finished products

$F = \{f\}$ = set of product families

$J = \{j\}$ = set of processors

$S = \{s\}$ = set of production sites

Parameters

p_{is} = price of raw material $i \in I^{RM}$ at site s

FC_f = setup cost for family f

v_{ijs} = variable production cost for product $i \in I^{RM}$ on processor j at site s

h_{is} = inventory holding cost for product $i \in I^{FP}$ at site s

ζ_{is} = penalty for dipping below target safety stock level of product $i \in I^{FP}$ at site s

$t_{iss'}/t_{is}$ = transportation cost to move a unit of product i from site s to site s'/customer location

μ_i = revenue per unit of product $i \in I^{FP}$

R_{ijs} = rate of production of product $i \in I^{RM}$ on processor j at site s

$\beta_{i'is}$ = the yield adjusted amount of raw material or intermediate product $i \in I^{FP}$ that must be consumed to produce a unit of $i' \in I^{RM}$ at site s

H_{js} = amount of time available for production on processor j at site s

λ_{if} = binary parameter indicating whether product i belongs to family f

MRL_{fjs} = minimum run length for family f on processor j at site s

θ_i = uncertain demand for product $i \in I^{FP}$

I^0_{is} = inventory of product i at site s at the start of the planning horizon

I^L_{is} = safety stock target for product i at site s

Variables

$$Y_{fjs} = \begin{cases} 1 & \text{if family } f \text{ is processed on processor } j \text{ at site } s \\ 0 & \text{otherwise} \end{cases}$$

P_{ijs} = production amount of $i \in I^{RM}$ on processor j at site s

RL_{ijs} = run length of product $i \in I^{RM}$ on processor j at site s

FRL_{fjs} = run length for family f on processor j at site s

C_{is} = consumption of raw material or intermediate product $i \in I^{FP}$ at site s

$W_{iss'}$ = flow of intermediate product $i \in I^{IP}$ from site s to s'

S_{is} = supply of finished product $i \in I^{FP}$ from facility s to the customer location

I_{is} = inventory level for $i \in I^{RM}$ at site s

I^-_i = amount of shortage of finished product $i \in I^{FP}$ at the customer location

I^Δ_{is} = deviation below target safety stock level for product $i \in I^{RM}$ at site s

Appendix A

Consider the LP dual of problem IP_{MPS}. By associating nonnegative dual variables u, v_s, and w_s with eqs 33, 34, and 35, respectively, this can be formulated as

$$\max_{u,v_s,w_s} -\theta u + \sum_{s\in S}(I^L_s - A_s)v_s - \sum_{s\in S} A_s w_s$$

subject to

$$u + v_s + w_s \geq \mu - t_s + h_s$$

$$v_s \leq \zeta_s$$

$$u,\ v_s,\ w_s \geq 0$$

The following dual solution is postulated for the three demand regimes:

Low-Demand Regime

$$u = \mu - \gamma_{s^*_l}$$

$$v_s = \begin{cases} \zeta_s & s \in ID \\ \gamma_{s^*_l} - \gamma_s & s \leq s^*_l - 1,\ \ s \in IS \\ 0 & s \geq s^*_l \quad,\ \ s \in IS \end{cases}$$

$$w_s = 0$$

Intermediate-Demand Regime

$$u = \mu - \omega_{s^*_l}$$

$$v_s = \begin{cases} \zeta_s & s \leq s^*_l \\ \zeta_s & s \geq s^*_l + 1,\ \ s \in ID \\ \omega_{s^*_i} - \gamma_s & s \geq s^*_l + 1,\ \ s \in IS \end{cases}$$

$$w_s = \begin{cases} \omega_{s^*_l} - \omega_s & s \leq s^*_l + 1 \\ 0 & s \geq -\gamma_s \end{cases}$$

High-Demand Regime

$$u = 0$$

$$v_s = \zeta_s$$

$$w_s = \mu - \omega_s$$

The feasibility of this solution is established by substituting into the constraints of the dual problem and using the cost assumptions (eqs 40−42) and the ranking schemes in the various demand regimes. In addition, this dual feasible solution results in the same objective function values for the three demand regimes as the postulated primal feasible solutions. Dual solution feasibility and equality of primal and dual objectives implies the optimality of the postulated supply policies based on the principle of strong LP duality.

Appendix B

Let $\alpha^L_{s^*_l}$, $\alpha^I_{s^*_l}$, and α^H be the probabilities that the product demand lies in the low-, intermediate-, and high-demand regimes, respectively.

$$\alpha^L_{s^*_l} = \Pr[\sum_{\substack{s=1\\ s\in IS}}^{s^*_l-1}(A_s - I^L_s) \leq \theta \leq \sum_{\substack{s=1\\ s\in IS}}^{s^*_l}(A_s - I^L_s)] \quad (66)$$

$$\alpha^I_{s^*_l} = \Pr[\Theta^{L\rightarrow I} + \sum_{\substack{s=1\\ s\in ID}}^{s^*_i-1} A_s + \sum_{\substack{s=1\\ s\in IS}}^{s^*_i-1} I^L_s \leq \theta \leq \Theta^{L\rightarrow I} + \sum_{\substack{s=1\\ s\in ID}}^{s^*_i} A_s + \sum_{\substack{s=1\\ s\in IS}}^{s^*_i} I^L_s] \quad (67)$$

$$\alpha^H = \Pr[\theta \geq \Theta^{I\rightarrow H}] \quad (68)$$

Subsequently, the recourse function $Q(A_s)$, which quantifies the expected second-stage costs for the multisite case, is given by

$$Q(A_s) = \mu\theta^m + \sum_{s\in S} h_s A_s + E^L(A_s) + E^I(A_s) + E^H(A_s) \quad (69)$$

where

$$E^L(A_s) = \sum_{s_l^* \in IS} \alpha^L_{s_l^*} E_\theta \left[\begin{array}{l} - [\mu - \gamma_{s_l^*}]\theta + \sum_{s \in ID} \zeta_s [I_s^L - A_s] \\ + \sum_{\substack{s=1 \\ s \in IS}}^{s_l^*-1} [\gamma_{s_l^*} - \gamma_s] [I_s^L - A_s] \\ | \sum_{\substack{s=1 \\ s \in IS}}^{s_l^*-1} (A_s - I_s^L) \le \theta \le \sum_{\substack{s=1 \\ s \in IS}}^{s_l^*} (A_s - I_s^L) \end{array} \right] \quad (70)$$

$$E^I(A_s) = \sum_{s_i^* \in IS \cup ID} \alpha^I_{s_i^*} E_\theta \left[\begin{array}{l} - [\mu - \omega_{s_i^*}]\theta + \zeta_{s_i^*} [I^L_{s_i^*} - A_{s_i^*}] \\ - \sum_{\substack{s=1 \\ s \in ID \cup IS}}^{s_i^*-1} [\omega_{s_i^*} - \omega_s] A_s + \sum_{\substack{s=1 \\ s \in ID \cup IS}}^{s_i^*-1} \zeta_s [I_s^L - A_s] \\ + \sum_{\substack{s \ge s_i^*+1 \\ s \in ID}} \zeta_s [I_s^L - A_s] + \sum_{\substack{s \ge s_i^*+1 \\ s \in IS}} [\omega_{s_i^*} - \gamma_s] [I_s^L - A_s] \\ | \Theta^{L \to I} + \sum_{\substack{s=1 \\ s \in ID}}^{s_i^*-1} A_s + \sum_{\substack{s=1 \\ s \in IS}}^{s_i^*-1} I_s^L \le \theta \le \Theta^{L \to I} + \sum_{\substack{s=1 \\ s \in ID}}^{s_i^*-1} A_s + \sum_{\substack{s=1 \\ s \in IS}}^{s_i^*-1} I_s^L \end{array} \right] \quad (71)$$

and

$$E^H(A_s) = \sum_{s \in S} \alpha^H E_\theta [-(\mu - \omega_s)A_s + \zeta_s (I_s^L - A_s) \mid \theta \ge \Theta^{I \to H}] \quad (72)$$

represent the expected second-stage costs in the low-, intermediate-, and high-demand regimes, respectively. Analytical integration of eqs 70–72, facilitated by the standardization of the demand parameter and the definition of $K^1_{s_l^*}$ and $K^2_{s_l^*}$ as

$$\sigma K^1_{s_l^*} = [\sum_{\substack{s=1 \\ s \in IS}}^{s_l^*-1} (A_s - I_s^L) - \theta^m]$$

$$\sigma K^2_{s_l^*} = [\Theta^{L \to I} + \sum_{\substack{s=1 \\ s \in ID}}^{s_i^*} A_s + \sum_{\substack{s=1 \\ s \in IS}}^{s_i^*} I_s^L - \theta^m] \quad (73)$$

results in

$$\begin{aligned} Q(A_s) = & \sum_{s_l^*=1}^{|IS|-1} \sigma[\gamma_{s_l^*+1} - \gamma_{s_l^*}][K^1_{s_l^*}\Phi(K^1_{s_l^*}) + f(K^1_{s_l^*})] \\ & + \sigma[\omega_{s_l^*=1} - \gamma_{s_l^*=|IS|}][K^1_{s_l^*=|IS|}\Phi(K^1_{s_l^*=|IS|}) + f(K^1_{s_l^*=|IS|})] \\ & + \sum_{s_i^*=1}^{|S|-1} \sigma[\omega_{s_i^*+1} - \omega_{s_i^*}] [K^2_{s_i^*}\Phi(K^2_{s_i^*}) + f(K^2_{s_i^*})] \\ & + \sigma[\mu - \gamma_{s_i^*=|S|}] [K^2_{s_i^*=|S|}\Phi(K^2_{s_i^*=|S|}) + f(K^2_{s_i^*=|S|})] \\ & - \sigma\mu K^2_{s_i^*=|S|} + \sum_{s \in S} \zeta_s I_s^L + \sum_{s \in S} t_s A_s \end{aligned} \quad (74)$$

where $|IS|$ denotes the IS site with the highest OSS cost and $|S|$ represents the site (IS or ID) with the highest USS cost. The convexity of the nonlinear terms in eq 74 is preserved because they retain the same form as the single-site case.

Literature Cited

(1) Subrahmanyam, S.; Pekny, J. F.; Reklaitis, G. V. Design of batch chemical plants under market uncertainty. *Ind. Eng. Chem. Res.* **1994**, *33*, 2688.

(2) Shah, N. Single and Multisite Planning and Scheduling: Current Status and Future Challenges. Foundations of Computer Aided Process Operations Conference (FOCAPO), Snowbird, Utah, July 5–10, 1998.

(3) Petkov, S. B.; Maranas, C. D. Multiperiod Planning and Scheduling of Multiproduct Batch Plants under Demand Uncertainty. *Ind. Eng. Chem. Res.* **1997**, *36*, 4864.

(4) Johns, W. R.; Marketos, G.; Rippin, D. W. T. The optimal design of chemical plant to meet time-varying demands in the presence of technical and commercial uncertainty. *Trans. Inst. Chem. Eng.* **1978**, *56*, 249.

(5) Reihhart, H. J.; Rippin, D. W. T. Design of flexible multiproduct plants – A new procedure for optimal equipment sizing under uncertainty. American Institute of Chemical Engineers Meeting, New York, 1987.

(6) Wellons, H. S.; Reklaitis, G. V. The design of multiproduct batch plants under uncertainty with staged expansion. *Comp. Chem. Eng.* **1989**, *13*, 115.

(7) Shah, N.; Pantelides, C. C. Design of multipurpose batch plants with uncertain production requirements. *Ind. Eng. Chem. Res.* **1992**, *31*, 1325.

(8) Ierapetritou, M. G.; Pistikopoulos, E. N. Batch plant design and operations under uncertainty. *Ind. Eng. Chem. Res.* **1996**, *35*, 772.

(9) Straub, D. A.; Grossmann, I. E. Integrated stochastic metric of flexibility for systems with discrete states and continuous parameter uncertainties. *Comp. Chem. Eng.* **1990**, *14*, 967.

(10) Straub, D. A.; Grossmann, I. E. Design optimization of stochastic flexibility. *Comp. Chem. Eng.* **1993**, *17*, 339.

(11) Georgiadis, M. C.; Pistikopoulos, E. N. An integrated framework for robust and flexible process systems. *Comp. Chem. Eng.* **1999**, *38*, 133.

(12) Liu, M. L.; Sahinidis, N. V. Optimization in Process Planning under Uncertainty. *Ind. Eng. Chem. Res.* **1996a**, *35*, 4154.

(13) Clay, R. L.; Grossmann, I. E. A disaggregation algorithm for the optimization of stochastic planning models. *Comp. Chem. Eng.* **1997**, *21*, 751.

(14) Liu, M. L.; Sahinidis, N. V. Robust process planning under uncertainty. *Ind. Eng. Chem. Res.* **1998**, *37*, 1883.

(15) McDonald, C.; Karimi, I. A. Planning and Scheduling of Parallel Semicontinuous Processes. 1. Production Planning. *Ind. Eng. Chem. Res.* **1997**, *36*, 2691.

(16) Liu, M. L.; Sahinidis, N. V. Long Range Planning in the Process Industries: A Projection Approach. *Comp. Oper. Res.* **1996**, *23*, 237.

(17) Petkov, S. B.; Maranas, C. D. Design of Single-Product Campaign Batch Plants under Demand Uncertainty. *AIChE J.* **1998**, *44*, 896.

(18) Nahmias, S. *Production and Operations Analysis*; Richard D. Irwin Inc.: Homewood, IL, 1989.

(19) Dantzig, G. B. Linear Programming Under Uncertainty. *Manage. Sci.* **1955**, *1*, 197.

(20) Ierapetritou, M. G.; Pistikopoulos, E. N.; Floudas, C. A. Operational Planning under Uncertainty. *Comp. Chem. Eng.* **1994**, *S18*, S553.

(21) Ierapetritou, M. G.; Pistikopoulos, E. N. Simultaneous incorporation of flexibility and economic risk in operational planning under uncertainty. *Comp. Chem. Eng.* **1994a**, *18*, 163.

(22) Ierapetritou, M. G.; Pistikopoulos, E. N. Novel Optimization Approach of Stochastic Planning Models. *Ind. Eng. Chem. Res.* **1994**, *33*, 1930.

(23) Pistikopoulos, E. N. Uncertainty in process design and operations. *Comp. Chem. Eng.* **1995**, *S19*, S553.

(24) Pistikopoulos, E. N.; Ierapetritou, M. G. Novel approach for optimal process design under uncertainty. *Comp. Chem. Eng.* **1995**, *19*, 1089.

(25) Clay, R. L.; Grossmann, I. E. Optimization of Stochastic Planning Models. *Trans. Inst. Chem. Eng.* **1994**, *72*, 415.

(26) Birge, J. R.; Louveaux, F. *Introduction to Stochastic Programming*; Springer: New York, 1997.

(27) Dantzig, G. B.; Wolfe, P. The decomposition principle for linear programs. *Oper. Res.* **1960**, *8*, 101.

(28) Benders, J. F. Partitioning procedures for solving mixed-variables programming problems. *Numer. Math.* **1962**, *4*, 238.

(29) Diwekar, U. M.; Kalagnanam, J. R. Efficient sampling technique for optimization under uncertainty. *AIChE J.* **1997**, *43*, 440.

(30) Ierapetritou, M. G.; Acevedo, J.; Pistikopoulos, E. N. An optimization approach for process engineering problems under uncertainty. *Comp. Chem. Eng.* **1996**, *20*, 703.

(31) Acevedo, J.; Pistikopoulos, E. N. Stochastic optimization based algorithms for process synthesis under uncertainty. *Comp. Chem. Eng.* **1998**, *22*, 647.

(32) Gupta, A.; Maranas, C. D. A hierarchical Lagrangean relaxation procedure for solving midterm planning problems. *Ind. Eng. Chem. Res.* **1999**, *38*, 1937.

(33) Acevedo, J.; Pistikopoulos, E. N. A Hybrid Parametric/Stochastic Programming Approach for Mixed-Integer Linear Problems under Uncertainty. *Ind. Eng. Chem. Res.* **1997**, *36*, 2262.

(34) Gal, T. *Postoptimal Analyses, Parametric Programming, and Related Topics*; Walter de Gruyter: Berlin, Germany, 1995.

(35) Geoffrion, A. M. Generalized benders decomposition. *J. Optim. Theory Appl.* **1972**, *10*, 237.

(36) Duran, M. A.; Grossmann, I. E. An outer approximation algorithm for a class of mixed-integer nonlinear programs. *Math. Prog.* **1986**, *36*, 307.

(37) Balas, E. Disjunctive programming and a hierarchy of relaxations for discrete optimization problems. *SIAM J. Alg. Discuss. Methods* **1985**, *6*, 466.

Received for review December 30, 1999
Accepted July 22, 2000

IE9909284